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NONLINEAR CHARACTERISTICS OF THE INTERACTION OF A TURBULENT BOUNDARY LAYER WITH A WAVY SURFACE

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A nonlinear interaction of a turbulent boundary layer with a wavy surface of a solid body or a liquid whose level has a deviation in the form of a traveling monochromatic wave is studied. For the waviness of small curvature, a calculation procedure is proposed for the amplitude dependences of the drag coefficient and complex elasticity which characterizes the back action of the flow on the surface inflection. The analysis is based on the use of an isotropic algebraic model of turbulent viscosity and an orthogonal system of curvilinear coordinates that follow the surface inflections. The interaction between the flow and the surface wave is described within the framework of a quasi-linear model, and a two-scale mean-flow model is used to determine the transverse structure of the flow in a smoothly expanding boundary layer.

When elastic coatings interact with a boundary layer, traveling shear waves that significantly affect the drag force can be generated on the surface of these coatings. Linear instabilities induced by the interaction of elastic coatings with a laminar boundary layer were studied in detail [1, 2]. Nonlinear effects were also considered for the case of a laminar flow [3, 4]. At the same time, much attention was also paid in experiments to the turbulent flow regime [5, 6]. The flow structure in a turbulent boundary layer over wavy surfaces was studied in some papers (see, for example, [7]), and the drag coefficient was determined. To derive evolution equations that describe the development of instability of the surface waves, however, it is necessary to know nonlinear characteristics of the interaction between an oscillating flow and these waves.

The study of the back action of a turbulent boundary layer on the wave motion of the surface is of interest for laboratory modeling of the waves on a water surface with a turbulent wind [8]. As in the case of the flow over coatings, a boundary layer of finite thickness which is matched with an external potential flow is formed above the water surface under laboratory conditions. In this case, the question arises whether the laboratory modeling is adequate to the response of the atmospheric boundary layer whose velocity profile is usually assumed to be logarithmic at an arbitrary distance from the surface.

In the present work, we study nonlinear characteristics of the interaction between a turbulent boundary layer of an incompressible flow and a wavy surface of a solid body or a liquid. The analysis is based on the approach developed by Jenkins [9] and Reutov and Troitskaya [10] for the description of the nonlinear interaction of water waves with an atmospheric (logarithmic) boundary layer. This approach includes the use of an orthogonal system of curvilinear coordinates, the hypothesis of isotropic turbulent viscosity, and a quasi-linear approximation in solving the equations of an oscillating flow. As in [10], the hydrodynamics equations are written in terms of the stream function and vorticity, which allows a significant speed-up of the convergence of the iteration process. To describe a smooth expansion of the boundary layer in the downstream direction, we use an approximation similar to that proposed previously for the calculation of boundary layers on smoothly curved airfoils [11, p. 459]. This allows us to construct the solution of the problem for a wavy surface on the basis of the theory of an equilibrium (self-similar) boundary layer that arises over a flat surface [12].

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Governing Equations. Quasi-Linear Approximation. We consider a turbulent boundary layer over a wavy surface. The deviation of the level of this surface in the y direction of the Cartesian system of coordinates (x, y) follows the law $\zeta = a\cos[k(x-ct)]$, where a is the waviness amplitude and k and c are the wavenumber and the phase velocity of the traveling wave. The surface curvature is assumed to be small, and the downstream expansion of the boundary layer is assumed to be smooth: $ka \ll 1$, $1/(kL) \ll 1$, and $\delta/L \ll 1$, where δ is the boundary-layer thickness and $L = \delta/(d\delta/dx)$ is the scale of boundary-layer expansion. For large values of y the boundary layer is assumed to transform to a uniform flow with a streamwise velocity $U(x)$, and the equation for the longitudinal component of momentum is $UdU/dx = -dP/dx$, where P is the pressure normalized to the flow density. Using the hypothesis of isotropic turbulent viscosity to close the turbulent flow equations, we introduce the effective (total) viscosity normalized to the density of the liquid, $\nu = \nu_0 + \nu_t$, where ν_0 and ν_t are molecular and turbulent viscosities.

Following [9, 10] we pass to the following system of orthogonal curvilinear coordinates (ξ, η) that moves with the phase velocity of the surface wave:

$$x - ct = \xi - a e^{-k\eta}\sin k\xi; \tag{1a}$$

$$y = \eta + a e^{-k\eta}\cos k\xi. \tag{1b}$$

The coordinate line $\eta = 0$ follows the surface deviations to an accuracy of the terms $\sim ka$. The basic assumption about the turbulent viscosity is that it is sufficient to consider explicitly only its dependence on the "transverse" coordinate η. Then the equations of two-dimensional hydrodynamics for the stream function Φ and vorticity χ determined in the moving frame of reference take the form [10]

$$\begin{gathered}\frac{\partial(\chi J^3)}{\partial t} + J^2[\Phi_\eta\chi_\xi - \Phi_\xi\chi_\eta - (\nu\chi)_{\xi\xi} - (\nu\chi)_{\eta\eta}] = -2J\nu_{\eta\eta}\Phi_{\xi\xi} - J_\eta((\Phi_\eta\nu_\eta)_\eta - \nu_\eta\Phi_{\xi\xi}) \\ -J_\xi(2\nu_\eta\Phi_{\xi\eta} - \Phi_\xi\nu_{\eta\eta}) + \Phi_\eta\nu_\eta\frac{J_\xi^2 + J_\eta^2}{J}, \qquad \Phi_{\xi\xi} + \Phi_{\eta\eta} = J\chi,\end{gathered} \tag{2}$$

where $J = \partial(x,y)/\partial(\xi,\eta) = 1 - 2ka e^{-k\eta}\cos k\xi + (ka)^2 e^{-2k\eta}$ is the Jacobian of mapping of the coordinates.

On a wavy surface for system (2) we impose the condition of continuity of the normal component of the velocity, which vanishes in the moving frame of reference, and the "no-slip" condition. Within the framework of the quasi-linear approximation used in the sequel, it is sufficient to write the boundary conditions to an accuracy of the terms $\sim ka$, as was done by Benjamin [13]. We can ignore the tangential component of the velocity on the rigid surface and prescribe this component in accordance with the potential theory of gravitational capillary waves on the liquid surface. Taking into account that there is an induced flow with velocity $v_0 \ll c$ on the liquid surface, we obtain

$$\Phi_\eta = -c + \begin{pmatrix}0\\ v_0\end{pmatrix} + \begin{pmatrix}1\\ 2\end{pmatrix} cka\cos k\xi, \qquad \frac{\partial\Phi}{\partial\xi} = 0\Big|_{\eta=0}. \tag{3}$$

Hereinafter the upper and lower coefficients in brackets refer to the solid and liquid surfaces, respectively.

We seek the solution of system (2) in a quasi-linear approximation which relates the nonlinear effects of the waviness amplitude only with the mean-flow deformation [9, 10]:

$$\Phi = \Phi_0(\eta, x) + [(1/2)\Phi_1(\eta, x)e^{ik\xi} + \text{c.c.}], \qquad \chi = \chi_0(\eta, x) + [(1/2)\chi_1(\eta, x)e^{ik\xi} + \text{c.c.}], \tag{4}$$

where Φ_0 and χ_0 are the mean (over the waviness period) components of the stream function and vorticity, Φ_1 and χ_1 are the complex amplitudes of the first harmonic, and c.c. refers to a complex conjugate expression. To describe the process of a smooth (with a scale $\sim L$) downstream expansion of the turbulent boundary layer, we introduce into (4) a smooth dependence of the mean and oscillating flow on the coordinate x related to ξ, η, and t by Eq. (1a). For fixed x and η, the variable $-\xi/c$ plays the role of a "time lag." After substituting (4) into (2), the derivatives with respect to t, ξ, and η, taking into account (1a), become

$$\frac{\partial}{\partial t} \to c\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial\xi} \to \frac{\partial}{\partial\xi} + (1 - ka e^{-k\eta}\cos k\xi)\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial\eta} \to \frac{\partial}{\partial\eta} + ka e^{-k\eta}\sin k\xi\frac{\partial}{\partial x}. \tag{5}$$

We note that the effective viscosity also depends on x; hence, strictly speaking, the derivatives of ν with respect to ξ should be retained in Eqs. (2). As is seen from (5), however, this would add small terms (with the derivatives of ν with respect to x) that can be ignored.

Substituting (4) and (5) into (2), we separate out the component $\sim\exp ik\xi$ in system (2). It can be easily seen that the flow "nonparallelism" yields allowances of the order of $O[a/L, (\delta/L)ka]$. Confining ourselves to an accuracy of $\sim ka$, we obtain a system of equations derived by Reutov and Troitskaya [10] for a parallel flow. To describe the mean flow, in contrast to [10], we introduce the stream function in the motionless frame of reference $\Psi = \Phi + cy$ which, in accordance with (1b), has the mean component $\Psi_0 = \Phi_0 + c\eta$ relative to ξ. As a result, system (10) from [10] takes the form[1]

$$ik[(\Psi_{0\eta} - c)\chi_1 - \Phi_1\chi_{0\eta}] - \left(\frac{d^2}{d\eta^2} - k^2\right)(\nu\chi_1) = 2\nu_{\eta\eta}k^2\Phi_1 - 2k^2a\,e^{-k\eta}[(\Psi_{0\eta} - c)\nu_\eta]_\eta,$$
$$\left(\frac{d^2}{d\eta^2} - k^2\right)\Phi_1 = \chi_1 - 2ka\chi_0\,e^{-k\eta}. \qquad (6)$$

Since there are no terms with the derivative $\partial/\partial x$ in (6), the oscillating flow is actually defined within the framework of the locally parallel theory.

We separate out the mean component relative to ξ in Eqs. (2) and compare them with equations obtained in [10] for a parallel mean flow. If we confine ourselves to an accuracy of $\sim\delta/L$ and, consequently, cancel the allowances $O[k^2a^2(\delta/L), ka^2/L]$, additional terms are only those with the derivative $\partial/\partial x$ that arise in the theory of the boundary layer on a flat surface [14].

After these transformations, Eq. (2) is integrated one time with respect to η, and an arbitrary function x which results from the integration is found using the above-mentioned equation for the longitudinal component of momentum in the external flow. As a result, we obtain the following equations for the stream function Ψ_0 and vorticity χ_0 of the mean flow in a motionless coordinate system:

$$\Psi_{0\eta}\frac{\partial}{\partial x}\Psi_{0\eta} - \Psi_{0\eta\eta}\frac{\partial}{\partial x}\Psi_0 = -\frac{dP}{dx} + (\nu\chi_0)_\eta - F_1; \qquad (7a)$$

$$\Psi_{0\eta\eta} = \chi_0 + F_2, \qquad (7b)$$

where $F_1 = k^2a\nu_\eta\mathrm{Re}\,(\Phi_{1\eta} - k\Phi_1)\,e^{-k\eta} + 2k(ka)^2\nu_\eta(\Psi_{0\eta} - c)\,e^{-2k\eta} - (1/2)k\,\mathrm{Im}\,(\Phi_1^*\chi_1)$ and $F_2 = \chi_0(ka)^2\,e^{-2k\eta} - ka(\mathrm{Re}\,\chi_1)\,e^{-k\eta}$. For $\partial/\partial x = 0$ Eqs. (7) coincide with those obtained in [10] for a parallel flow, and for $F_{1,2} = 0$ they transform to the Prandtl equations for a turbulent boundary layer on a flat plate [14]. With account of the transformations (5), the boundary conditions (3) on the wetted surface are

$$\Psi_{0\eta} = \begin{pmatrix} 0 \\ v_0 \end{pmatrix}, \qquad \frac{\partial}{\partial x}\Psi_0 = 0, \qquad \Phi_1 = 0, \qquad \Phi_{1\eta} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} cka\bigg|_{\eta=0}. \qquad (8)$$

According to (6) and (7), in the potential flow region ($\chi_0 = \chi_1 = 0$) the solution has the form $\Psi_{0\eta} = U(x)$ and $\Phi_1 \sim \exp(-k\eta)$. Assuming that at a certain level $\eta = \delta_1 > \delta$ the difference between the flow under study and the potential flow can be ignored, we write the external boundary conditions as

$$\Psi_{0\eta} = U, \qquad \chi_1 = 0, \qquad \Phi_{1\eta} + k\Phi_1 = 0\bigg|_{\eta=\delta_1}. \qquad (9)$$

It should be noted that the mean-flow deformation under the condition of waviness has an order of $(ka)^2$. If we seek the mean flow as an expansion of the perturbation theory relative to $ka \ll 1$, the mean-flow deviation from the profile in the case of a flat surface yields the terms $\sim(ka)^3$ in Eq. (6). At the same time, the quasi-linear approximation does not consider the contribution of the second harmonic of an oscillating flow, which also leads to terms of the order of $(ka)^3$ in (6). Thus, a sequential use of the perturbation theory relative to ka contradicts, at first glance, the quasi-linear approximation. This issue was studied in [10] in considering the interaction of water waves with the atmospheric boundary layer. It was found that the relative variation of the growth rate calculated in the quasi-linear approximation can be represented as the product of

[1]In Eqs. (6) we corrected the misprints made in [10].

$(ka)^2$ and a large numerical coefficient, whereas the contribution of the second harmonic is represented as the product of $(ka)^2$ and a numerical coefficient not greater than unity. This numerical result can be explained by the fact that the profile of the second harmonic is more susceptible to oscillations relative to η than the profile of the first (fundamental) harmonic. Without additional justification, it is assumed in the present work that the presence of a rather large coefficient at $(ka)^2$ in the relative increment to the growth rate and the drag coefficient ensures the applicability of the quasi-linear approximation.

Basic Parameters of the Flow. We introduce the displacement thickness for the boundary layer on a wavy surface δ^* as the thickness of the displaced layer of the potential flow relative to the η coordinate:

$$\delta^* = \int_0^{\delta_1} \left(1 - \frac{u_0}{U}\right) d\eta, \tag{10}$$

where $u_0 = \Psi_{0\eta}$ is the profile of the "effective" velocity determined from the mass-flow rate. In deriving formula (10), it is taken into account that we can assume $u_0 = U$ in the potential flow to an accuracy of the terms $\sim(ka)^2$. Ignoring the flow nonparallelism, we obtain a relation of the form $v_1 - c = (\Phi_\eta x_\xi - \Phi_\xi x_\eta)/J$ for the x-component of the velocity in the motionless frame of reference. Since the relative allowances to the mean flow are severalfold greater than $(ka)^2$ within the limits of applicability of the quasi-linear approximation, it is possible to ignore the terms that contain $(ka)^2$ in an explicit form when calculating v_1. We obtain

$$\langle v_1 \rangle = u_0(\eta) + (1/2)\, ka\, e^{-k\eta}\, \mathrm{Re}\,(\Phi_{1\eta} + k\Phi_1), \tag{11}$$

where $\langle \ldots \rangle$ refers to the mean value over the period relative to ξ. As is seen from (9) and (11), in the potential flow region u_0 coincides with $\langle v_1 \rangle$.

To determine the drag coefficient c_f we use the known expression for the x-component of the force acting per unit area of the wetted surface [11]: $T_1 = -pn_1 + \sigma_{11}n_1 + \sigma_{12}n_2$, where p is the surface pressure, $\sigma_{11} = 2\nu\Psi_{xy}$ and $\sigma_{12} = \nu(\Psi_{yy} - \Psi_{xx})$ are the components of the viscous stress tensor, and $n_{1,2}$ are the components of the normal vector to the wetted surface (the quantities T_1, p, σ_{11}, and σ_{12} are normalized to the flow density). Taking into account that the length element along the coordinate line $\eta = 0$ is $J^{1/2} d\xi$, we write the force per unit area of the undisturbed surface $y = 0$ as

$$\frac{1}{2} c_f U^2 = \frac{1}{\lambda} \int_0^{\lambda} T_1 J^{1/2}\, d\xi = \langle p y_\xi - \sigma_{11} y_\xi + \sigma_{12} y_\eta \rangle \Big|_{\eta=0} \tag{12}$$

($\lambda = 2\pi/k$ is the wavelength). We use a procedure described in [10] for surface pressure calculation. This allows us to represent the complex amplitude of the first harmonic p_1 in the form

$$p_1 = -i\,\frac{\nu_0}{k}\,\chi_{1\eta} + \begin{pmatrix}0\\1\end{pmatrix} c^2 ka \Big|_{\eta=0}. \tag{13}$$

Calculating σ_{11} and σ_{12} in the curvilinear coordinates ξ and η and retaining the terms that contain explicitly only the first power of ka, we write the drag coefficient as

$$c_f = \frac{2}{U^2}\left[\nu_0 \chi_0 - \frac{1}{2}\,\nu_0 ka\, \mathrm{Re}\,(k^{-1}\chi_{1\eta} + \chi_1)\right]_{\eta=0}. \tag{14}$$

We prescribe the turbulent viscosity as was done by Mellor and Gibson [12] for the boundary layer on a flat surface. We assume a linear profile of turbulent viscosity $\nu_t = æ u_* \eta$ which begins outside the buffer region ($\eta^+ = \eta u_*/\nu_0 > 30$) and reaches a constant value $\nu_t = KU\delta^*$ (u_* is the dynamic velocity, æ is the Kàrmàn constant, equal to 0.4, and K is the Clauser constant, equal to 0.016). Smoothing the resultant inflection by the function tanh and using one of the known approximations for the profile $\nu(\eta)$ in the buffer region [15], we obtain

$$N = \frac{\nu}{U\delta^*} = \frac{1}{\mathrm{R}} + K \tanh\left(\frac{æ}{K} Y\right)[1 - \exp(-\mu Y^2 \mathrm{R}^2)], \tag{15}$$

where $\mathrm{R} = \delta^* U/\nu_0$ is the Reynolds number, μ is a constant factor, and $Y = \eta/\Delta$ ($\Delta = \delta^*/\gamma$ is the displacement thickness relative to the velocity defect and $\gamma = u_*/U$). In the absence of waviness, the linear profile $\nu(\eta)$ corresponds to the logarithmic profile of the velocity [14]

$$\frac{u_0}{u_*} = \frac{1}{æ} \ln \eta^+ + B, \tag{16}$$

where B is a universal constant. If the viscosity is defined by Eq. (15), the law-of-the-wall (16) is approximately fulfilled in the domain $30 < \eta^+ < (K/æ)\,\mathrm{R}$. The upper boundary of this domain corresponds to the point of inflection in the piecewise-linear model of viscosity. Calculations of the logarithmic profile of the velocity, which were conducted on the basis of formula (15) with $K \to \infty$, showed that the known value of the universal constant $B = 5.0$ is reached for $\mu = 0.0019$.

To determine the dynamic velocity u_* in the boundary layer on a wavy surface, we have to find the longitudinal component of the viscous–turbulent flux of momentum through the external boundary of the buffer region $\eta^+ = 30$. Using the corresponding law of conservation (see Eq. (2) from [10]) and canceling the terms that contain $(ka)^2$ in an explicit form, we obtain

$$u_*^2 = \langle \sigma_{11} x_\eta + \sigma_{12} x_\xi \rangle \Big|_{\eta^+=30} = \nu\chi_0 - \frac{1}{2}\nu k a\, e^{-k\eta}\, \mathrm{Re}\,\chi_1 \Big|_{\eta^+=30}. \tag{17}$$

Note that, in the boundary layer on a wavy surface, generally speaking, u_*^2 can differ from the drag force.

We characterize the back action of an oscillating flow on the surface waves by a complex elasticity G [16, 17]. Its imaginary part describes the flux of energy S from the boundary layer to the surface wave:

$$G = \frac{\tilde{p}_1}{a}, \qquad S = -\langle \tilde{p}\zeta_t \rangle = \frac{1}{2} c k a^2\, \mathrm{Im}\, G. \tag{18}$$

Here $\tilde{p}_1$ is the complex amplitude of the effective surface pressure $\tilde{p}$. For a rigid surface, the flux of energy S is determined by the normal component of the surface stress and we should assume $\tilde{p} = -(n_1 T_1 + n_2 T_2)$, where T_2 is the y-component of the surface force per unit area. Calculations accurate to the terms $\sim ka$ show that the viscous normal stress makes no contribution to $\tilde{p}$, i.e., $\tilde{p}_1 = p_1$ to this accuracy. In accordance with (13) and (18) we obtain

$$\mathrm{Im}\, G = -\frac{\nu_0}{ka}\, \mathrm{Re}\, \chi_{1\eta} \Big|_{\eta=0}. \tag{19}$$

In an air flow over the water surface the growth rate of the surface waves is determined by the action of both normal and tangential stresses. Tangential stresses transform to normal stresses due to modulation of a dynamic boundary layer that arises near the water surface. Exactly this total normal stress enters Eq. (18). It is found by solving the equations of the dynamic boundary layer in water under the condition of continuity of σ_{ij} at the interface and has the form [9] $\tilde{p}_1 = p_1 - \hat{\sigma}_{22} + i\hat{\sigma}_{12}$ ($\wedge$ denotes the complex amplitude of the first harmonic). Retaining the terms $\sim ka$ in determining $\tilde{p}$, we obtain

$$\mathrm{Im}\, G = -\frac{\nu_0}{ka}\, \mathrm{Re}\,(\chi_{1\eta} - k\chi_1) + 4\nu_0 c k^2 - 2\nu_0 k \chi_0 \Big|_{\eta=0}. \tag{20}$$

Below we solve the problem of calculation of the imaginary part of the dimensionless complex elasticity $g = \mathrm{Im}\, G/(kU^2)$, which, according to (18), is also a normalized flux of energy to the surface wave.

Two-Scale Model for Mean-Flow Calculation. We assume that a uniform (self-similar) flow regime is realized in the external region of the boundary layer (the "wake" region) in the absence of waviness. A theory of such a boundary layer was constructed by Mellor and Gibson [12]. An alternative to the solution of equations in partial derivatives (6) for the boundary layer on a wavy surface is the determination of the profiles of the stream function, velocity, and vorticity of the mean flow on the basis of a two-scale approach constructed by analogy with the theory of the equilibrium boundary layer. A similar analogy was used previously to calculate laminar boundary layers on smoothly curved surfaces [11].

Following [12], we choose the displacement thickness Δ relative to the velocity defect as the main (external) scale of the mean flow and introduce a dimensionless pressure gradient $b = (\delta^*/u_*^2)dP/dx$. In addition, we assume $\alpha = k\Delta$ and $C = c/u_*$ and introduce a parameter $q = d\Delta/dx$ that characterizes the boundary-layer expansion. The solution of system (6) and (7) with the boundary conditions (8) and (9) is sought in the form

$$\Psi_0 = -u_* \Delta f(Y) + U\eta, \qquad \chi_0 = \frac{u_*}{\Delta}\,\Omega_0(Y); \tag{21a}$$

$$\Phi_1 = \varphi_1 u_* \Delta, \qquad \chi_1 = \Omega_1 u_*/\Delta, \tag{21b}$$

where $Y = \eta/\Delta(x)$ is the dimensionless "transverse" coordinate used in (15). We note that the derivative $f_Y = (U - u_0)/u_*$ determines the velocity profile defect u_0. Substituting (21) into (7), assuming that $u_* = \gamma U$, and ignoring the contribution of $d\gamma/dx$, we obtain the equations for the mean flow:

$$\frac{d}{dY}\,N\Omega_0 = \Big(\frac{q}{\gamma} - b\Big)(Y f_{YY} - \gamma f f_{YY}) + b(2f_Y - \gamma f_Y^2) + \mathcal{F}_1; \tag{22a}$$

$$\frac{d^2 f}{dY^2} = -(\Omega_0 + \mathcal{F}_2), \tag{22b}$$

where

$$\mathcal{F}_1 = \alpha\,(ka)\,N_Y\,\mathrm{Re}\,(\varphi_{1Y} - \alpha\varphi_1)\,e^{-\alpha Y} + 2\alpha\,(ka)^2 N_Y \bar{u}_0\, e^{-2\alpha Y} - (1/2)\,\alpha\,\mathrm{Im}\,(\varphi_1^* \Omega_1),$$
$$\mathcal{F}_2 = \Omega_0 (ka)^2\, e^{-2\alpha Y} - ka\, e^{-\alpha Y}\,\mathrm{Re}\,\Omega_1 \quad (\bar{u}_0 = 1/\gamma - f_Y - C)).$$

For $Y = 0$ and $Y = \delta_1/\Delta$, the following boundary conditions should be valid for the mean-flow fields:

$$f = 0, \qquad f_Y = \frac{1}{\gamma} - \frac{v_0}{u_*}\Big|_{Y=0}, \qquad N\Omega_0 - \frac{1}{2}\,kaN\,\mathrm{Re}\,(\alpha^{-1}\Omega_{1Y} + \Omega_1) = \frac{2c_f}{\gamma^2}\Big|_{Y=0}; \tag{23a}$$

$$f_Y = 0, \qquad f = 1\Big|_{Y=\delta_1/\Delta}. \tag{23b}$$

Relation (17), which determines the dynamic velocity, takes the form

$$N\Omega_0 - \frac{1}{2}\,N\,ka\mathrm{Re}\,\Omega_1\, e^{-\alpha Y} = 1\Big|_{Y=30/\mathrm{R}}. \tag{24}$$

Apart from the boundary conditions that follow from (8) and (9), Eqs. (23) include dimensionless analogs of relations (10) and (14) which also have the form of the boundary conditions. For $a = 0$, system (22) reduces to the equation for f derived in [12] for an equilibrium boundary layer on a flat surface. In contrast to [12], we ignore the derivative $d\gamma/dx$, which yields allowances of a higher order for "flow nonparallelism," since the scale of variation of γ is much greater than L.

After passing to dimensionless variables, system (6) for an oscillating flow takes the form

$$i\alpha[\bar{u}_0\Omega_1 - \varphi_1\Omega_Y] = \Big(\frac{d^2}{dY^2} - \alpha^2\Big)(N\Omega_1) + 2N_{YY}\alpha^2\varphi_1 - 2\alpha ka\, e^{-\alpha Y}\,\frac{d}{dY}\,[N_Y\bar{u}_0],$$
$$\Big(\frac{d^2}{dY^2} - \alpha^2\Big)\varphi_1 = \Omega_1 - 2ka\, e^{-\alpha Y}\Omega_0. \tag{25}$$

The boundary conditions (8) and (9) yield

$$\Omega_1 = 0, \qquad \varphi_{1Y} + \alpha\varphi_1 = 0\Big|_{Y=\delta_1/\Delta}, \qquad \varphi_1 = 0, \qquad \varphi_{1Y} = \Big(\frac{1}{2}\Big)\,cka\Big|_{Y=0}. \tag{26}$$

Though only one scale Δ was explicitly used in the derivation of Eqs. (22), the mean-flow velocity profile actually has two scales, since the viscosity (15) defines a viscous scale $\nu_0/u_* = \Delta/\mathrm{R}$ in the wall region. Equations (22), however, are also valid near the surface, since the contribution of the flow nonparallelism terms ($\sim q$, b) is small by virtue of rapid changes in the fields with respect to Y. In fact, the equations for the wall region coincide with the relations obtained in [10] for a parallel flow in which the longitudinal component of the mean total flux of momentum with respect to ξ through the coordinate lines η = const is rigorously retained. In a weakly nonparallel flow this component changes only slightly in the wall region and, according to (14), equals $(1/2)c_f U^2$ on the surface $\eta = 0$. In the presence of waviness the total flux of momentum can be divided into the wave component and the mean-flow contribution. Therefore, the behavior of the mean-velocity profile in the wall region depends substantially on the variation of the wave field.

Numerical Procedure and Calculation Results. The solution of the boundary-value problem (22)–(26) should be constructed so that the prescribed "input" parameters R, b, ka, α, and C determined the unknown "output" parameters γ, q, and c_f. Prior to the numerical solution, we substituted the variables $z = \ln(Y + Y_0)$, where the parameter Y_0 controls the diminution of the step along Y (with a fixed step along z). The boundary-value problem (25) and (26) for oscillating fields was solved by the method of iterations in which the distributions of the mean fields were taken from the previous iteration. Derivatives were replaced by finite differences, and the Gauss elimination technique was applied to the resultant difference equations with a fixed step along z (see [10]). Integration of Eqs. (22) from $Y = 0$ to $Y = \delta_1/\Delta$ with the initial data (23a) determines f, $f_Y|_{Y=\delta_1/\Delta}$, and the left-hand side of relation (24) as functions of the parameters γ, q, and c_f, which allows us to consider conditions (23b) and (24) as a system of three algebraic equations relative to γ, q, and c_f. This system was solved by the Newton method in which the derivatives with respect to the sought variables were replaced by finite differences (a three-dimensional variant of the secant method). The mean-flow equations (22) were integrated with respect to z using the second-order Runge–Kutta method.

The main calculations were performed for $Y_0 = 5 \cdot 10^{-4}$ and $\delta_1/\Delta = 0.5$ on a grid composed of 500 equidistant points in the z direction. For a zero pressure gradient ($b = 0$) the boundary-layer thickness δ determined from the level $u_0 = 0.99U$ was roughly equal to half of the interval of integration with respect to Y ($\delta/\Delta \simeq 0.25$). Its value increased for negative pressure gradients ($b < 0$); however, the ratio δ/Δ approached δ_1/Δ only for b close to its extreme possible value [12] $b = -0.5$. The effect of flow deviation from the potential flow at the end of this interval was verified in selective calculations with $\delta_1/\Delta = 1$ at 1000 points. The convergence of the iteration procedure became worse as ka increased. No more than 10 to 15 iterations were made for $ka = 0.1$. The Newton method for the boundary-value problem (22)–(24) converged usually after 2–4 iterations.

The nonlinear characteristics of the flow depend on the conditions of increasing amplitude. Within the framework of the numerical procedure, the value of ka increased for constant parameters R, $b_1 = (\delta^*/U^2)dP/dx = \gamma^2 b$, $k\delta^* = \gamma\alpha$, and $c/U = \gamma C$ which, in contrast to the above set of the input parameters, are defined via the "external" scales of the flow U and δ^*. The choice of the displacement thickness δ^* as an external parameter was determined, in particular, by the fact that the derivative $d\delta^*/dx$ defines the slope of the mean-flow streamlines outside the boundary layer. This allows us to assume that the displacement thickness is a factor that exhibits the least changes in passing from a smooth sector of the surface to a wavy sector. For small ka the growth rates of the parameters of the interaction between the boundary layer and waviness are quadratic in amplitude:

$$g = g_0 + g_1(ka)^2, \qquad \gamma = \gamma_0 + \gamma_1(ka)^2, \qquad c_f = c_{f0} + (ka)^2 c_{f1}, \quad \text{etc.} \tag{27}$$

The coefficients in expansions (27) are independent of a and determine the linear and nonlinear features of the flow. It is easily seen that $c_{f0} = 2\gamma_0^2$.

In calculations of the interaction of a turbulent boundary layer with a long gravitational wave on the water surface, the wavenumber and the boundary-layer thickness were chosen in accordance with the data of [8]: $k = 0.04$ cm^{-1} and $\delta \sim 25$ cm (the phase velocity was $c = 1.57$ m/sec). The drift flow velocity in water v_0 was assumed equal to u_* [18]. Figure 1 shows the coefficients g_0 and g_1 versus the flow velocity in the boundary layer above the gravitational wave (in Fig. 1a, curves 1′ and 2′ refer to c_{f1}, curves 1 and 1′ refer to $\delta^* = 4$ cm, curves 2 and 2′ refer to $\delta^* = 8$ cm, the solid curves correspond to $b_1 = 0$, and the dashed curves refer to $b_1 = -0.0003$). The value $b_1 = -0.0003$ is a significant negative pressure gradient in the range of U/c values considered, since the parameter b reaches half of its limiting value. The data obtained show a weak influence of the negative pressure gradient on the behavior of both linear and nonlinear parts of the complex elasticity.

If in the definition of c_f (14) and in the expression for $\operatorname{Im} G$ (20) we retain only the contribution of the surface pressure, we obtain $c_{f1} = g_0$. Therefore, the difference between g_0 and c_{f1} in Fig. 1a is determined by the effect of tangential surface stresses. The ratio δ/δ^* depends on δ_* and on the pressure gradient. In

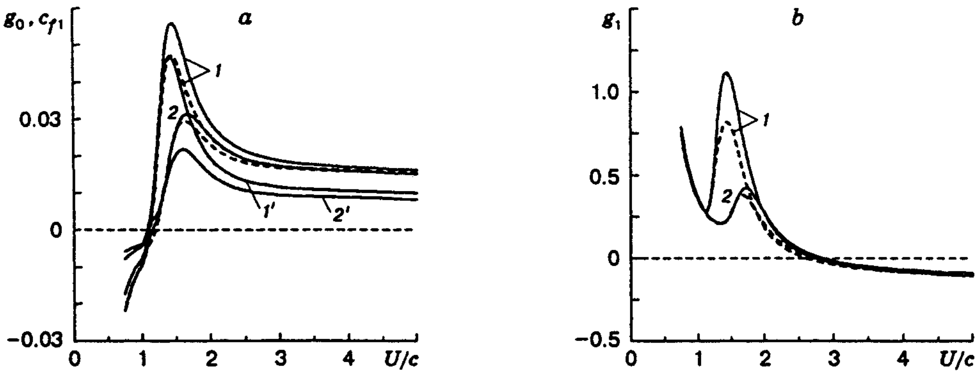

Fig. 1

particular, for the solid curve 1 this ratio increases monotonically from 6.5 to 7.5 as U/c increases. We can add that dependences similar to those plotted in Fig. 1 were obtained for $U = \text{const}$ and the variation of U/c owing to the variation of the wavenumber.

It is seen from Fig. 1 that for $U/c \leqslant 2.5$ the amplitude growth rate $\operatorname{Im} G$ becomes positive ($g_1 > 0$), which corresponds to a tough excitation of the wave. The quantity g_1 changes sign when the critical level determined by the condition $u_0(y) = c$ approaches the upper boundary of the buffer region of the boundary layer where $u_0 \simeq 0.55U$. A soft excitation was obtained in [10] for a logarithmic boundary layer of infinite thickness and the same values of k. To compare the dependence of the interaction coefficient β between the waves and the wind on u_*/c, which were derived in [10], with the theory of the boundary layer of finite thickness, we should assume $\beta = æ^2 g/\gamma^2$ and $u_*/c = \gamma U/c$. Thus, the behavior of the flux of energy to the surface waves under laboratory conditions can be different from that under full-scale conditions.

It is seen from Fig. 1 that an anomalous increase of g_0 and g_1 is observed near $U/c \approx 1.4$. The calculations showed that this effect is manifested only for rather long waves ($k\delta \leqslant 1$). The anomalous increase of the interaction parameters is intimately related to the finite thickness of the boundary layer, since for $U/c \simeq 1.4$ the critical level turns out to be near the "inflection" in the effective viscosity profile (15). A dramatic attenuation of anomalous deviations was registered in test calculations when the logarithmic sector of the velocity profile was extended by a factor of 1.5–2 due to an increase of the constant K in (15).

Similar calculations conducted for the coefficients γ_0 and γ_1 showed that the behavior of γ_0 corresponds to that known for the boundary layer on a flat surface. For example, for $\delta^* = 4$ cm and $b_1 = 0$, the coefficient γ_0 monotonically decreases from 0.039 to 0.033 within the range $0.75 < U/c < 5$ (Fig. 1). The nonlinear parameter γ_1 is negative, which corresponds to a decrease of u_* in the presence of waviness. This is in qualitative agreement with the data of [8]. In the vicinity of $U/c \simeq 1.4$ the coefficient γ_1 exhibits an anomalous deviation toward negative values. Thus, for $\delta^* = 4$ cm and $b_1 = 0$ we obtain $\gamma_1 = -0.048$ for $U/c = 5$ and $\gamma_1 = -0.14$ for $U/c = 1.4$.

The calculations of the velocity profile u_0 showed that, in the presence of waviness, the parameter u_0/U decreases within the entire interval of Y except for the boundary values 0 and 1. A buffer region, a logarithmic sector, and a wake region are clearly distinguished in the profile. This differs from the behavior of the velocity profile in a logarithmic boundary layer in which the decrease of the mean-flow velocity acquires a constant value with distance from the surface [10]. The calculations showed that the substitution of $\langle v_1 \rangle$ for u_0 [see (11)] practically does not alter the pattern of the mean-flow velocity-profile deformation induced by waviness. The substitution of $\langle v_1 \rangle$ for u_0 in determining the displacement thickness (10) also has only a weak effect on the results. The most significant decrease of the mean velocity is observed for $U/c \simeq 1.4$, i.e., in the region of anomalous increase of g_0 and g_1. This can be associated with the growth of nonlinearity resulting from a deeper penetration of the oscillating flow into the boundary layer.

To estimate the effect of waviness on the mean flow we can use the profile of the viscous–turbulent shear stress determined by the relation on the right-hand side of relation (17) taken with an arbitrary η.

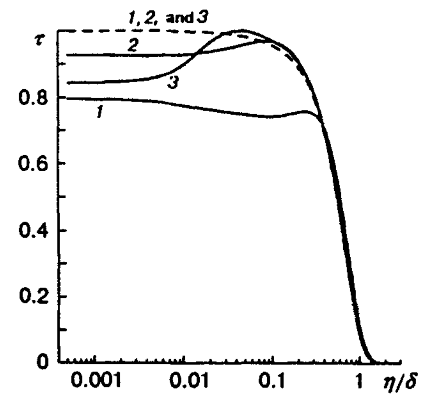

Fig. 2

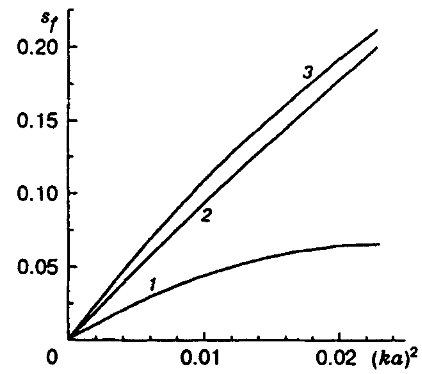

Fig. 3

The calculations showed that this shear stress is easily approximated by each of the three quantities $\nu u_{0\eta}$, $\nu \chi_0$, or $\nu \langle v_{1\eta} \rangle$, which allows us to consider the shear stress as the mean-flow contribution to the longitudinal component of the flux of momentum to the wavy surface. The ratio of the viscous–turbulent stress to the drag force (per unit area) defined as $\tau = 2\nu u_{0\eta}/(c_f U^2)$ is shown in Fig. 2 ($\delta^* = 4$ cm and $b_1 = 0$, curves 1 and 2 refer to $U/c = 1.4$ and 3, respectively, and the solid and dashed curves correspond to $ka = 0.1$ and 0.001, respectively). It is seen that for $ka \to 0$, when the wave portion of the flux of momentum vanishes, $\tau(0)$ is close to unity. For $ka = 0.1$ the value of $\tau(0)$ is noticeably smaller than unity, which can be explained by the emergence of a positive wave component of the flux of momentum.

The linear portion of the turbulent viscosity profile corresponds to the region $0.02 \leqslant \eta/\delta \leqslant 0.15$ for curves 1 or $0.02 \leqslant \eta/\delta \leqslant 0.15$ for curves 2. It follows from here that for $ka = 0.1$ the quantity τ varies comparatively weakly both in the region of existence of a linear profile of viscosity [where a logarithmic profile of the velocity (16) appears] and in the buffer region of the boundary layer. In this case, u_*^2 coincides with the viscous component of the drag force, which is significantly lower than its total value. This behavior of τ can be explained by weak changes in the wave flux of momentum at distances $\eta \ll \delta$ from the surface because of the large length of the surface wave ($k\delta \leqslant 1$).

Though the ratio u_0/U decreases in the presence of waviness, the constant B can increase due to a decrease of u_*. Thus, for $ka = 0.001$ the calculations predict $B \approx 6.0$, which exceeds the constant $B = 5.0$ for a flat surface by the value of the dimensionless velocity of the surface drift v_0/u_*. At the same time, the values $B \approx 6.0$ and 6.45 for $U/c = 3$ and 1.4, respectively, were obtained for $ka = 0.1$. The increase of the constant B is manifested much more noticeably in the experiments [8], which is possibly connected with the peculiarities of determination of u_*.

The calculations for a solid surface were performed for the case of slow surface waves ($c \ll U$) which under laboratory conditions are usually observed on viscous–elastic coatings [5, 6]. In this case, we can assume $c \to 0$, and the results of calculation of the drag coefficient are also applicable to a stiff wavy profile. Figure 3 shows the amplitude dependence of the relative growth rate of the drag coefficient $s_f = (c_f - c_{f0})/c_{f0}$ (R = 3200, $b_1 = 0$, curves 1–3 refer to $k\delta^* = 0.05$, 0.8, and 2.5, respectively). It is seen that the drag coefficient increases as the waviness amplitude increases, and deviations from the quadratic law are manifested at first for long waves. The behavior of the linear and nonlinear characteristics of the energy exchange between the boundary layer and a slow wave on a rigid coating versus the wavenumber is shown in Fig. 4 ($b_1 = 0$, curves 1–4 refer to R = 1500, 3200, 6000, and 9000). The energy inflow to the surface wave, which is determined by the parameter $g_0 > 0$ in Fig. 4a, is attenuated by the action of nonlinearity ($g_1 < 0$ in Fig. 4b). The contribution of the mean flow to the shear stress for a rigid surface with a comparatively small-scale waviness ($k\delta \approx 5$) is shown by curves 3 in Fig. 2 (R = 3200, $b_1 = 0$, $k\delta^* = 0.8$, the solid and dashed curves refer to $ka = 0.1$ and 0.001, respectively). The linear sector of the viscosity profile corresponds to the range $0.035 \leqslant \eta/\delta \leqslant 0.15$ in which the variation of τ is comparatively small and, hence, a logarithmic velocity profile (16) is realized. The behavior of τ shows that the attenuation of the wave portion of the flux of momentum, even for moderately

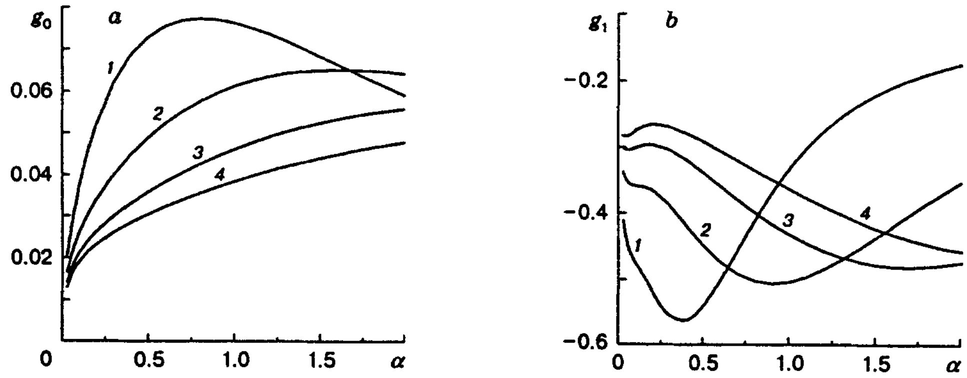

Fig. 4

large $k\delta$, occurs within the buffer region. As a result, u_*^2 coincides with the drag force and exceeds the force of viscous friction on the surface by a factor of $1/\tau(0)$.

To elucidate the difference between the characteristics of the interaction between the boundary layer and the surface of a liquid or a rigid coating, we compared the results of the solution of the problem with the wind over long waves on the water ($k\delta \sim 1$) without drift ($v_0 = 0$) and the problem of the flow over a rigid surface. No significant changes were observed for either linear or nonlinear characteristics of interaction.

Conclusion. Thus, a nonlinear interaction of long surface waves on the water with a boundary layer of finite thickness can be qualitatively different from the interaction of these waves with the atmospheric boundary layer (anomalous growth of the imaginary part of the complex elasticity and transition from soft to tough excitation). Nonlinear increments to the complex elasticity are small for $ka \ll 1$; nevertheless, they are of principal importance, for example, in derivation of the evolution equations for the surface waves near the threshold of their stability.

An explicit logarithmic sector of the mean-velocity profile in the boundary layer on a wavy surface arises in the case of long ($k\delta \leqslant 1$) and short ($k\delta \geqslant 5$) waves. The dynamic viscosity defined as a parameter of the logarithmic profile underpredicts the drag force for the large-scale waviness and the total value of the drag force for the small-scale waviness.

An important element of the proposed approach is the use of a two-scale model for mean-flow calculations. To evaluate the applicability of the model, we should note that the mean-flow variations caused by the waviness are in most cases localized on a small portion of the boundary layer near the surface. The flow evolution in the wall region is determined by the variation of the dynamic velocity (for constant amplitude of waviness, wavenumber, and phase velocity). Therefore, the changes introduced by the wall region are manifested in the wake flow, which has its own scale of establishment $\sim L$. In other cases, we can assume that this model gives reasonable estimates for nonlinear parameters of the flow response and the drag coefficient.

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